Lemma 1

Let f be an increasing concave function, such that $f(0) = 0$. Let's have $a, c \in \mathbb{R}^+$ such that $a < c$. Then

$$\frac{f(a)}{f(c)} > \frac{a}{c}$$

Proof: Consider a linear function

$$g(x) = \frac{f(c)}{c}x \tag{1}$$

According to this function, we get $g(a)$ which divides the interval $\langle 0, f(c)\rangle$ in the same proportion as a divides the interval $\langle 0, c\rangle$ (this is a direct consequence of $g(x)$'s linearity). By substitution of $x = a$ in (1) we get

$$\frac{g(a)}{f(c)} = \frac{a}{c} \tag{2}$$

Because $f(x)$ is concave, $g(x) < f(x), \forall x \in (0, c)$ (by definition of concave functions), hence,

$$g(a) < f(a) \tag{3}$$

Finally, by (2) and (3), it follows that

$$\frac{f(a)}{f(c)} > \frac{a}{c}$$

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